OSCILLATION OF A CASCADE OF THIN SECTIONS IN INCOMPRESSIBLE FLOW

(KOLEBANIIA RESHETKI TONKIKH PROFILEI V NESZHIMAEMOM POTOKE)

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The method of solution presented in [1] lends itself to the analysis of the perturbed fluid motion and the hydrodynamic forces due to the oscillation of thin cascades. Specifically, consider a cascade consisting of an inclined row of flat blades of width d and period $l \exp i\beta$ (see Fig. 1) ($i = \sqrt{-1}$). The study of the perturbed motion about such cascades is directly related to the problem of the flow in turbomachines. An approximate solution of the problem of small oscillations of the cascade, in which the effect of the oscillating cascade is represented by a system of concentrated vortices, is given in [2].

In what follows, we assume, that the time-dependence of the normal velocity v_n and the velocity potential of the fluid motion depends on the exponential factor $\exp j\sigma t$, where the imaginary index $j = \sqrt{-1}$ is to be distinguished from the imaginary index $i = \sqrt{-1}$ which appears in the variable $z = x + iy$. We separate the complex potential into two parts, $w_0(z)$ and $w_1(z)$, where $w_0(z)$ defines the non-circulatory flow about the cascade, and $w_1(z)$ is the solution of a homogeneous problem (cf. [1]). In the example under consideration, $w(z)$ is a periodic function with period $l \exp i\beta$, and the functions w_0 and w_1 satisfy on the blades the conditions

$$\operatorname{Im} \frac{dw_0}{dz} = -v_n, \qquad \operatorname{Im} w_1 = A \qquad (1)$$

where A is a constant, to be determined.

The oncoming flow with velocity v_0 approaches the cascade from the direction $x < 0$; thus the function $f(z) = r + is$ is related to w_1 through the relation (ref. [1])

$$f(z) = \frac{dw_1}{dz} + j\mu_0 w_1 \qquad \left(\mu_0 = \frac{\sigma}{v_0}\right) \qquad (2)$$

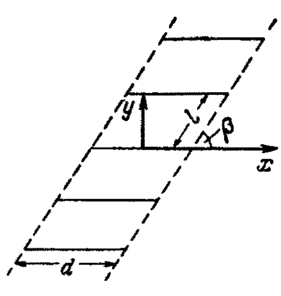

Fig. 1.

It follows, due to (1), that on the blades we have the condition

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$$\operatorname{Im} f = j\mu_0 A \tag{3}$$

Besides conditions (1) and (3) there exist the equations

$$\lim \frac{dw_0}{dz} = 0, \qquad \lim f = 0 \qquad \text{for } z_1 \to \infty \qquad \left(z_1 = z \exp\left[-i\left(\frac{1}{2}\pi + \beta\right)\right]\right) \tag{4}$$

which express the fact that ahead of the cascade the fluid is not perturbed.

It is known that the function

$$z = \frac{l}{\pi}\left(u \sin\beta - i\cos\beta\, \frac{\operatorname{ch} u + (\operatorname{sh}^2 u - \operatorname{sh}^2 a)^{1/2}}{\operatorname{ch} a}\right) \tag{5}$$

conformally maps the given cascade in the z-plane onto a vertical cascade in the u-plane, consisting of flat blades of width $2a$ and period πi. The edges of a blade in the z-plane correspond to points in the u-plane for which

$$\frac{dz}{du} = \frac{l}{\pi}\left(\sin\beta - i\cos\beta\, \frac{\operatorname{sh} u}{(\operatorname{sh}^2 u - \operatorname{sh}^2 a)^{1/2}}\right) = 0 \tag{6}$$

From relations (5) and (6) we have

$$\begin{gathered} u = \pm u_0, \qquad u_0 = ar\operatorname{sh}(\operatorname{sh} a \sin\beta) \\ d = \frac{2l}{\pi}\cdot[\sin\beta\, ar\operatorname{sh}(\operatorname{sh} a\sin\beta) + \cos\beta\, ar\sin(\operatorname{th} a\cos\beta)] \end{gathered} \tag{7}$$

In the u-plane the conditions (1) for the function w_0 transform to conditions of the following form on the segments $(-a, a)$.

$$\operatorname{Im}\frac{dw_0}{du} = -v_n \frac{dz}{du} \tag{8}$$

We represent the function dw_0/du in the form of a sum $F_1 + F_2$; then it follows from (8) that on the segments $(-a, a)$ the functions F_1 and F_2 satisfy the conditions

$$\operatorname{Im} F_1 = \pm v_n \frac{l}{\pi}\frac{\cos\beta \operatorname{sh} u}{(\operatorname{sh}^2 a - \operatorname{sh}^2 u)^{1/2}}, \qquad \operatorname{Im} F_2 = -v_n \frac{l}{\pi}\sin\beta \tag{9}$$

where the plus sign indicates an approach to the segment $(-a, a)$ from above, while the minus sign indicates an approach to the segment from below.

It is clear that $F_1(u)$ represents a velocity potential function which gives a distribution of sources on the segments $(-a, a)$, while $F_2(u)$ is the velocity potential function which gives the effect of a distribution of vortices along these segments. It follows that for the non-circulatory flow it is necessary, apart from conditions (4), that $\operatorname{Im} F_2(u) = 0$ for $u \to \infty$. Applying the methods of the theory of cascades [3], we obtain the following expressions for the function w_0 :

$$\frac{dw_0}{du} = \frac{l}{\pi^2}\cos\beta \int_{-a}^{a} \frac{v_n \operatorname{sh}\xi}{(\operatorname{sh}^2 a - \operatorname{sh}^2\xi)^{1/2}}[\operatorname{cth}(\xi - u) - 1]\,d\xi + \tag{10}$$

$$+\frac{l\sin\beta}{\pi^2 i\,(\mathrm{sh}^2 u-\mathrm{sh}^2 a)^{1/2}}\int\limits_{-a}^{a}\frac{v_n(\mathrm{sh}^2 a-\mathrm{sh}^2\xi)^{1/2}}{\mathrm{sh}\,(\xi-u)}\,d\xi$$

In particular, for pure translational oscillations, we obtain from formula (10),

$$\frac{dw_0}{du}=\frac{l}{\pi}v_n e^{-i\beta}\left(1-\frac{\mathrm{sh}\,u}{(\mathrm{sh}^2 u-\mathrm{sh}^2 a)^{1/2}}\right) \tag{11}$$

In an analogous way, the function f is determined from conditions (3) and (4). We have

$$f=\left\{-\frac{l}{\pi}j\mu_0 A e^{-i\beta}\left[1-\frac{\mathrm{sh}\,u}{(\mathrm{sh}^2 u-\mathrm{sh}^2 a)^{1/2}}\right]-\frac{\Gamma e^{u}}{2\pi i\,(\mathrm{sh}^2 u-\mathrm{sh}^2 a)^{1/2}}\right\}\frac{du}{dz} \tag{12}$$

where Γ is a real constant (with respect to i) which, as will be seen in what follows, represents the complex amplitude of the circulation around the blade.

For the determination of the constants Γ and A we have the condition that the velocity be finite at the trailing edges of the cascade, and condition (1) for the function w_1. Satisfying the condition of finite velocity at the trailing edge, i.e. for $z = d/2 (u = u_0)$, we obtain the relations

$$\begin{aligned}\Gamma=\Gamma_0\Bigg\{&\frac{\cos^2\beta}{\pi}\int\limits_{-a}^{a}\frac{v_n\,\mathrm{sh}\,\xi}{(\mathrm{sh}^2 a-\mathrm{sh}^2\xi)^{1/2}}[\mathrm{cth}\,(\xi-u_0)-1]\,d\xi-\\&-\frac{\sin\beta}{\pi\,\mathrm{sh}\,a}\int\limits_{-a}^{a}\frac{v_n(\mathrm{sh}^2 a-\mathrm{sh}^2\xi)^{1/2}}{\mathrm{sh}\,(\xi-u_0)}\,d\xi-j\mu_0 A\Bigg\}\end{aligned} \tag{13}$$

$$\Gamma_0=2l\,\mathrm{sh}\,a\,(\mathrm{sh}\,a\sin n\beta-(1+\mathrm{sh}^2 a\sin^2\beta)^{1/2}) \tag{14}$$

For pure translational hearing oscillations, (13) has the much simpler form

$$\Gamma=\Gamma_0(v_n-j\mu_0 A)=2l\,\mathrm{sh}\,a\,(\mathrm{sh}\,a\sin\beta-(1+\mathrm{sh}^2 a\sin^2\beta)^{1/2})(v_n-j\mu_0 A) \tag{15}$$

We will not satisfy condition (1) for the function w_1. Since the flow is not perturbed far ahead of the cascade, in the direction perpendicular to its axis, we have, from (2),

$$w_1=e^{-j\mu_0 z}\int\limits_{(L)}^{z}e^{j\mu_0 z}f(z)\,dz \tag{16}$$

where the integration in (16) is taken along a certain line which joins z and a point which is infinitely far ahead of the cascade and in a direction perpendicular to its axis.

Taking into consideration condition (1), we find, from (16),

$$A = e^{j\nu} D_y \qquad \left(\nu = \frac{\mu_0 d}{2}\right), \qquad D_x + i D_y = \int\limits_{(L)}^{-d/2} e^{j\mu_0 z} f(z)\, dz \tag{17}$$

To carry the calculations further, let us examine a contour consisting of two semi-infinite straight lines, perpendicular to the axis of the cascade and joining, at their ends, the segment $(-d/2,\ -d/2 + l \exp i\beta)$. It is evident that the integral of the function $f(z) \exp j\mu_0 z$ over this contour is equal to zero. Therefore, using the periodicity of the function $f(z)$ and also relation (12), we will have

$$D_x + i D_y = -\frac{1}{1 + e^{j\mu_0 \tau}} \int\limits_{-u_0}^{-u_0+\pi i} e^{j\mu_0 z(u)} F(u)\, du \qquad (\tau = l e^{i\beta}) \tag{18}$$

$$F(u) = -\frac{l}{\pi} j\mu_0 A e^{-i\beta}\left(1 - \frac{\operatorname{sh} u}{(\operatorname{sh}^2 u - \operatorname{sh}^2 a)^{1/2}}\right) - \frac{\Gamma e^{u}}{2\pi i (\operatorname{sh}^2 u - \operatorname{sh}^2 a)^{1/2}} \tag{19}$$

Introducing a change of variable $u = -u_0 + i\xi$ and making use of the identity

$$D_y = \frac{1}{2j}\left[(D_x + jD_y) - (D_x - jD_y)\right]$$

we obtain the following relation

$$A = \Gamma K_0 - j\mu A K_1 \qquad \left(\mu = \mu_0 \frac{l}{\pi}\right) \tag{20}$$

Here K_0 and K_1 represent functions of the parameters ν, d/l and β, and are determined by integral expressions (the bar on a letter denotes the complex conjugate of the quantity, with respect to j)

$$K_0 = -\frac{1}{4\pi} e^{j\nu}\left[\frac{1}{1 - e^{j\mu\delta}} \int\limits_0^{\pi} \exp(-u_0 + j\xi - j\theta + j\mu\zeta(\xi)) \frac{d\xi}{p(\xi)} + \right.$$
$$\left. + \frac{1}{1 - e^{j\mu\bar{\delta}}} \int\limits_0^{\pi} \exp(-u_0 - j\xi + j\theta + j\mu\bar{\zeta}(\xi)) \frac{d\xi}{p(\xi)}\right] \tag{21}$$

$$K_1 = -\frac{1}{2} e^{j\nu}\left[\frac{e^{-j\beta}}{1 - e^{j\mu\delta}} \int\limits_0^{\pi}\left(1 - j\,\frac{sh(u_0 - j\xi) - j\theta}{p(\xi)}\, e^{-j\theta}\right) e^{j\mu\zeta(\xi)}\, d\xi + \right.$$
$$\left. + \frac{e^{j\beta}}{1 - e^{j\mu\bar{\delta}}} \int\limits_0^{\pi}\left(1 + j\,\frac{\operatorname{sh}(u_0 + j\xi)}{p(\xi)}\, e^{j\theta}\right) e^{j\mu\bar{\zeta}(\xi)}\, d\xi\right] \tag{22}$$

$$\delta = \pi e^{j\beta}, \quad \zeta(\xi) = (j\xi - u_0)\sin\beta - j\cos\beta \ln\frac{\operatorname{ch}(u_0 - j\xi) + jp(\xi)e^{j\theta}}{\operatorname{ch} a}$$
$$p^4(\xi) = \operatorname{sh}^4 a \cos^4\beta + \sin^4\xi + 2\operatorname{sh}^2 a \sin^2\xi(\cos^2\beta + 2\sin^2\beta \operatorname{ch}^2 a) \tag{23}$$
$$\sin 2\theta = \frac{\operatorname{sh} a \sin\beta \sin 2\xi}{p^2(\xi)}(1 + \operatorname{sh}^2 a \sin^2\beta)^{1/2}$$

In the given form, K_0 and K_1 may be fairly easily computed with the help of ordinary methods of numerical integration.

From relations (14) we find the final expressions:

$$\Gamma = \Gamma_s C\left(\nu, \frac{d}{l}, \beta\right), \qquad j\mu_0 A = \frac{\Gamma_s}{\Gamma_0}\left[1 - C\left(\nu, \frac{d}{l}, \beta\right)\right] \tag{24}$$

Here Γ_s represents the circulation for $\nu = 0$, that is, the value of the circulation which is determined under the hypothesis of stationarity

$$\Gamma_s = \Gamma_0 \left\{ \frac{\cos^2\beta}{\pi} \int_{-a}^{a} \frac{v_n \operatorname{sh}\xi}{(\operatorname{sh}^2 a - \operatorname{sh}^2\xi)^{1/2}} \left[\operatorname{cth}(\xi - u_0) - 1 \right] d\xi - \right.$$

$$\left. - \frac{\sin\beta}{\pi \operatorname{sh} a} \int_{-a}^{a} \frac{v_n (\operatorname{sh}^2 a - \operatorname{sh}^2\xi)^{1/2}}{\operatorname{sh}(\xi - u_0)} d\xi \right\} \qquad (25)$$

which, for $v_n = \text{const}$, is determined by a very simple expression, as may be seen from (15). The function $C(\nu,\ d/l,\ \beta\ )$ accounts for the non-stationary effects occurring during oscillation of the cascade, and has the form

$$C\left(\nu,\ \frac{d}{l},\ \beta\right) = \frac{1 + j\mu K_1}{1 + j\mu\,(K_1 + (\pi/l)\Gamma_0 K_\theta)} \qquad (26)$$

We will also determine the magnitude of the hydrodynamic force existing on a blade of the cascade. For this, we use the formula [1]

$$Y = -\rho v_0 e^{j\sigma t}\,\mathrm{Re} \int_{L_1} \left(\frac{dw_0}{dz} + j\mu_0 w_0 + f \right) dz \qquad (27)$$

where L_1 is a contour enclosing the blade and taken counter-clockwise.

Expression (27) may be divided into two parts:

$$Y = Y_0 + Y_1, \qquad Y_0 = -\rho v_0 e^{j\sigma t}\,\mathrm{Re} \int_{L_1} \left(\frac{dw_0}{dz} + j\mu_0 w_0 \right) dz, \qquad Y_1 = -\rho v_0 e^{j\sigma t}\,\mathrm{Re} \int_{L_1} f\,dz \qquad (28)$$

Using formulas (12) and (24), we find the following expression:

$$Y_1 = \rho v_0 \Gamma_s C\left(\nu,\ \frac{d}{l},\ \beta\right) e^{j\sigma t} \qquad (29)$$

The force Y_0 is determined by the non-circulatory motion of the fluid, and is related to the effect of apparent mass. For pure translational oscillations we have

$$Y_0 = -\mu_{yy} \frac{dV_n}{dt} \qquad \left(\mu_{yy} = \frac{2\rho l^2}{\pi} \ln \operatorname{ch} a,\ V_n = v_n e^{j\sigma t} \right) \qquad (30)$$

Existing computations [3] show that over a significant range of the solidity ratio d/l, the coefficient of apparent mass μ_{yy} depends only slightly on the inlet angle β.

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